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Normalized momentum maps and reduction

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Abstract. It is shown that for a Hamiltonian action of a compact Lie group on a symplectic manifold, a reduction scheme can be constructed based on a suitably normalized momentum map. Under the appropriate regularity assumptions it turns out that the reduced spaces are Poisson manifolds or, more precisely, canonical manifolds. As an application, the reduction of a particular class of non-Hamiltonian systems with symmetry is considered.

1. Introduction

In some recent papers, Leach [1] and Gorringe and Leach [2–4], have studied the conservation laws and orbit equations for a certain model of the Kepler problem with drag, and generalizations of it. An important element in their analysis is the fact that the angular momentum of the system under consideration yields a conformal invariant. It then immediately follows that the unit vector in the direction of the angular momentum is a vector constant of the motion, which enables one to reduce the equations of motion. The observation that conformal invariance of the angular momentum for the Kepler problem with drag can be related to the rotational symmetry of the system, has inspired us to investigate, from a more general perspective, the interplay between symmetry and reduction for a class of non-Hamiltonian systems admitting a conformal invariant induced by the symmetry group (cf [5]).

The theory of symmetry and reduction constitutes one of the most beautiful and most important chapters in the geometrical treatment of mechanics: see, e.g., Marmo *et al* [6]. We now briefly describe the two main cornerstones on which the whole theory is built, namely the Marsden–Weinstein reduction theorem for symplectic manifolds and (a geometric version of) Noether's theorem. For details we may refer to various textbooks, such as Abraham and Marsden [7], Guillemin and Sternberg [8], Libermann and Marle [9], and Marmo *et al* [6].

Let (M, ω) be a symplectic manifold, G a Lie group with Lie algebra $\mathcal{G}$, and $\mathcal{G}^*$ the linear dual space of $\mathcal{G}$. Assume G defines a symplectic action on (M, ω) with an Ad^*-equivariant momentum map $J : M \longrightarrow \mathcal{G}^*$, where Ad^* is the coadjoint representation of G on $\mathcal{G}^*$. For any $\mu \in \mathcal{G}^*$, let G_μ denote the isotropy subgroup of G at μ with respect to the coadjoint representation. The Marsden–Weinstein reduction theorem then states the following (see also [10]): if $\mu \in \mathcal{G}^*$ is a weakly regular value of J and if G_μ acts freely and properly on the submanifold $J^{-1}(\mu)$ of M, then there exists a unique symplectic structure ω_μ on the orbit space $\mathcal{P}_\mu = J^{-1}(\mu)/G_\mu$ such that $\pi_\mu^*\omega_\mu = j_\mu^*\omega$. Here, $\pi_\mu : J^{-1}(\mu) \longrightarrow \mathcal{P}_\mu$ denotes the canonical projection and $j_\mu : J^{-1}(\mu) \longrightarrow M$ is the natural inclusion. Following Libermann and Marle [9], a symplectic action will be called *Hamiltonian* if it admits a momentum

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map, and *strongly Hamiltonian* if the momentum map is Ad*-equivariant. In passing, we recall that for any symplectic action with momentum map J, there always exists an affine action of G on $\mathcal{G}^*$ with respect to which J is equivariant [11]. The reduction theorem can therefore be formulated for every Hamiltonian action, whether or not the associated momentum map is Ad*-equivariant [9].

Within the above geometrical setting Noether's theorem can be formulated as follows: if a Hamiltonian action of G on (M, ω), with momentum map J, leaves invariant a Hamiltonian system, then J is a conserved quantity (i.e. a $\mathcal{G}^*$-valued first integral) of that system. Using both theorems it is then not difficult to prove that a Hamiltonian system with symmetry induces a reduced Hamiltonian system on the symplectic quotient spaces $\mathcal{P}_\mu$, parametrized by the weakly regular values of the momentum map.

The Marsden–Weinstein theorem has been extended and generalized in several ways by relaxing some of the technical assumptions. In that respect we just refer here to a treatment of reduction in the case of non-free group actions [12] and the description of a general reduction procedure which also deals with singular values of the momentum map (see, e.g., [13, 14]). On the other hand, the framework for the reduction theory has also been extended from symplectic manifolds to general Poisson manifolds [15, 16] and to cosymplectic and contact manifolds [17]. (For the cosymplectic case, see also [18]).

Suppose we are given a strongly Hamiltonian action of a compact Lie group G on a symplectic manifold (M, ω), with Ad*-equivariant momentum map J. The purpose of this paper now is to establish a reduction scheme for M with respect to the *normalized momentum map* $\hat{J} = J/\|J\|$, where $\|\ \ \|$ is the norm induced by a G-invariant metric on $\mathcal{G}^*$. It will be shown that the reduced spaces corresponding to the weakly regular values of $\hat{J}$ are Poisson manifolds or even, under the appropriate assumptions, canonical manifolds in the sense of Lichnerowicz [19]. As pointed out before, the main motivation for this work stems from the study of the reduction of a class of non-Hamiltonian systems with symmetry. In particular, we are thinking here of the phase space description of certain types of mechanical systems with friction which admit a compact symmetry group, and for which it turns out that the normalized momentum map $\hat{J}$, rather than the momentum map itself, is a conserved quantity.

In section 2 we first briefly review the general reduction procedure for Poisson manifolds as described in [15]. This will then serve as the basic tool in deriving a reduction scheme for a symplectic manifold in terms of a normalized momentum map in section 3. In section 4 we will briefly describe how this construction can be applied to establish a reduction of a particular class of non-Hamiltonian systems with a compact symmetry group, defined on a cotangent bundle (see also [5]).

The treament is confined to finite-dimensional smooth manifolds and Lie groups. All maps, vector fields and differential forms are assumed to be of class C^∞.

2. Poisson reduction

Let $(P, \{\ ,\ \}_P)$ be a Poisson manifold. We denote the corresponding Poisson tensor field on P by Λ, i.e. Λ is a skew-symmetric contravariant 2-tensor field with vanishing Schouten bracket, and such that for any two smooth functions f and g on P, $\Lambda(df, dg) = \{f, g\}_P$ (see, e.g., [9]). Let $M \subset P$ be a submanifold of P with natural inclusion $i : M \longrightarrow P$. Assume we are given, at each point $m \in M$, a linear subspace E_m of T_mP. We then consider the 'generalized' sub-bundle E of $TP|_M$ defined by $E = \bigcup_{m\in M} E_m$.

Suppose now that E satisfies the following conditions:

(i) $E \bigcap TM$ is an integrable sub-bundle of TM, inducing a foliation $\mathcal{E}$ on M.
(ii) The foliation $\mathcal{E}$ is regular in the sense that the quotient space $M/\mathcal{E}$ is a smooth manifold and the projection $\pi : M \longrightarrow M/\mathcal{E}$ is a submersion.
(iii) If f, g are smooth functions on P with differentials vanishing on E, then $\mathrm{d}\{f, g\}_P$ also vanishes on E.

Under these conditions one can prove that $M/\mathcal{E}$ admits a Poisson structure if and only if

(iv) $\Lambda(E^0) \subset E + TM$

where E^0 is the annihilator of E and Λ is regarded as a vector bundle homomorphism from T^*P to TP (cf [15]). Moreover, the Poisson bracket $\{\ ,\ \}_{M/\mathcal{E}}$ on $M/\mathcal{E}$ is then defined by the relation

$$\{f, g\}_P \circ i = \{\tilde{f}, \tilde{g}\}_{M/\mathcal{E}} \circ \pi$$

for any two (locally defined) smooth functions $\tilde{f}, \tilde{g}$ on $M/\mathcal{E}$ and any two (locally defined) smooth extensions f, g of $\tilde{f} \circ \pi, \tilde{g} \circ \pi$, with differentials vanishing on E. Following Marsden and Ratiu [15] we will call a triple (P, M, E) satisfying the conditions (i) to (iv) *Poisson reducible.*

3. Normalized momentum maps and reduction

Hereafter we assume that (M, ω) is a connected symplectic manifold, G a connected compact Lie group, and $\Phi : G \times M \longrightarrow M$ a strongly Hamiltonian (left) action of G on M, with Ad*-equivariant momentum map $J : M \longrightarrow \mathcal{G}^*$. Ad*-equivariance of J means that $\mathrm{Ad}^*_g \circ J = J \circ \Phi_g$ for every $g \in G$ where, as usual, $\Phi_g = \Phi(g, .)$. The G-orbit of a point $m \in M$ will be denoted by $G.m$ and the isotropy subgroup of G at m by G_m. For any $\xi \in \mathcal{G}$, the corresponding fundamental vector field (or infinitesimal generator) of the given action is the vector field ξ_M on M, defined by

$$\xi_M(m) = \frac{\mathrm{d}}{\mathrm{d}t}\Phi(\exp(-t\xi), m)|_{t=0}\,.$$

(For the definitions of ξ_M and of the coadjoint action Ad*, we follow the conventions of [9]). In particular, the map $\xi \longrightarrow \xi_M$ then yields a Lie algebra homomorphism from $\mathcal{G}$ into the Lie algebra $\mathcal{X}(M)$ of vector fields on M. If $\mathcal{G}_m$ denotes the Lie algebra of G_m, one can prove that (cf [20])

$$\mathcal{G}_m = \{\xi \in \mathcal{G} | \xi_M(m) = 0\}\,.$$

Since Φ is a Hamiltonian action with momentum map J we know that for any $\xi \in \mathcal{G}$

$$i_{\xi_M}\omega = -\mathrm{d}\langle \xi, J\rangle$$

with $\langle\ ,\ \rangle$ denoting the natural pairing between $\mathcal{G}$ and $\mathcal{G}^*$. The real-valued function $J_\xi = \langle \xi, J\rangle$ on M is a Hamiltonian for the (global) Hamiltonian vector field ξ_M. We further recall the following basic properties of the momentum map [6, 7, 8, 9]. At each point $m \in M$ the kernel (Ker) and image (Im) of the tangent map of J are given by

$$\mathrm{Ker}(T_m J) = T_m(G.m)^\perp \qquad (1)$$
$$\mathrm{Im}(T_m J) = \mathcal{G}_m^0 \qquad (2)$$

where '$\perp$' denotes the ω-orthogonal complement and $\mathcal{G}_m^0$ is the annihilator of $\mathcal{G}_m$ in $\mathcal{G}^*$. Moreover the momentum map $J : M \longrightarrow \mathcal{G}^*$ is a Poisson morphism with respect to the Poisson structure on M, induced by ω, and the canonical Lie–Poisson structure on $\mathcal{G}^*$. The fundamental vector fields of the action Φ on M and of the coadjoint action on $\mathcal{G}^*$ are J-related, i.e.

$$TJ \circ \xi_M = \xi_{\mathcal{G}^*} \circ J \tag{3}$$

for every $\xi \in \mathcal{G}$.

Since G is assumed to be compact, it is well known that its Lie algebra $\mathcal{G}$, and likewise the dual space $\mathcal{G}^*$, can be equipped with a G-invariant positive definite metric (see, e.g., [20]). Let us denote the metric on $\mathcal{G}^*$ by $(\ ,\ )$ and G-invariance then means that for any two elements $\mu, \nu \in \mathcal{G}^*$ and for every $g \in G$

$$(\mathrm{Ad}_g^* \mu, \mathrm{Ad}_g^* \nu) = (\mu, \nu)\,. \tag{4}$$

This further implies that for every $\xi \in \mathcal{G}$

$$(\mathrm{ad}_\xi^* \mu, \nu) + (\mu, \mathrm{ad}_\xi^* \nu) = 0$$

and thus, in particular, for each $\nu \in \mathcal{G}^*$

$$(\mathrm{ad}_\xi^* \nu, \nu) = 0 \tag{5}$$

where ad^* stands for the coadjoint representation of $\mathcal{G}$ on $\mathcal{G}^*$. The metric induces a norm on $\mathcal{G}^*$ according to $\|\mu\| = (\mu, \mu)^{1/2}$. With respect to this norm we can then introduce the normalized momentum map

$$\hat{J} : M \backslash J^{-1}(0) \longrightarrow \mathcal{G}^*, m \longmapsto \hat{J}(m) = \frac{J(m)}{\|J(m)\|}$$

where $M\backslash J^{-1}(0)$ is the open submanifold of M obtained by deleting the zero level set of J. For simplicity we henceforth put $M\backslash J^{-1}(0) = M_0$. The restriction of ω on M_0 will still be denoted by ω and, clearly, (M_0, ω) is a symplectic submanifold of (M, ω). Note that, since $J^{-1}(0)$ is a G-invariant subset of M, the given action Φ restricts to a strongly Hamiltonian action of G on (M_0, ω), which we also denote by Φ, and its associated momentum map is the restriction of J on M_0.

Let $S_{\mathcal{G}^*}$ be the unit sphere in $\mathcal{G}^*$, i.e. $S_{\mathcal{G}^*} = \{\mu \in \mathcal{G}^* \mid \|\mu\| = 1\}$, and consider the fibration

$$\rho : \mathcal{G}^* \setminus \{0\} \longrightarrow S_{\mathcal{G}^*}, \nu \longmapsto \rho(\nu) = \frac{\nu}{\|\nu\|}\,.$$

Upon identifying the tangent space at a point $\nu \in \mathcal{G}^*$ with $\mathcal{G}^*$, one easily verifies that

$$\mathrm{Ker}(T_\nu \rho) = \{r\nu \mid r \in \mathbb{R}\}\,. \tag{6}$$

Since $S_{\mathcal{G}^*}$ is a regular submanifold of $\mathcal{G}^*$ and $\mathrm{Im}(\hat{J}) \subset S_{\mathcal{G}^*}$, one can regard the normalized momentum map as a smooth map from M_0 into $S_{\mathcal{G}^*}$ and we then have

$$\hat{J} = \rho \circ J|_{M_0}\,. \tag{7}$$

Hereafter $\hat{J}$ will be interpreted in this way.

Lemma 1. $\mathrm{Ker}(T_m \hat{J}) = \{v \in T_m M_0 \mid T_m J(v) = rJ(m) \text{ for some } r \in \mathbb{R}\}$.

Proof. Using the chain rule for tangent maps it follows from (7) that $T_m\hat{J} = T_{J(m)}\rho \circ T_m J$ at each point $m \in M_0$. The lemma is then an immediate consequence of (6). □

In view of (4) we have that $\| \mathrm{Ad}^*_g \mu \| = \|\mu\|$ from which we can infer, in particular, that the coadjoint action restricts to a smooth action of G on $S_{\mathcal{G}^*}$ which we also denote by Ad^*. Taking into account the assumed Ad^*-equivariance of J, it readily follows that

$$\hat{J} \circ \Phi_g = \mathrm{Ad}^*_g \circ \hat{J} \tag{8}$$

for every $g \in G$, i.e. $\hat{J}$ is equivariant with respect to the induced actions of G on M_0 and $S_{\mathcal{G}^*}$. Recall that G_μ denotes the isotropy subgroup of G at $\mu \in \mathcal{G}^*$ under the coadjoint action, and $\mathcal{G}_\mu$ its Lie algebra.

Lemma 2. For every $\mu \in \mathrm{Im}(\hat{J}) \subset S_{\mathcal{G}^*}$, $\hat{J}^{-1}(\mu)$ is an invariant subset for the action of G_μ. Moreover, at each point $m \in \hat{J}^{-1}(\mu)$ the following relation holds:

$$\mathrm{Ker}(T_m\hat{J}) \bigcap T_m(G.m) = T_m(G_\mu.m)\,.$$

Proof. Invariance of $\hat{J}^{-1}(\mu)$ under G_μ immediately follows from the equivariance property (8) of $\hat{J}$. Now, let $m \in \hat{J}^{-1}(\mu)$ and $v \in T_m(G.m)$, i.e. $v = \xi_{M_0}(m)$ for some $\xi \in \mathcal{G}$. For the second part of the lemma it then suffices to prove that $v \in \mathrm{Ker}(T_m\hat{J})$ if and only if $\xi \in \mathcal{G}_\mu$.

According to lemma 1, $v \in \mathrm{Ker}(T_m\hat{J})$ iff $T_m J(v) = (TJ \circ \xi_{M_0})(m) = rJ(m)$ for some $r \in \mathbb{R}$. Taking into account (3), this is still equivalent with

$$\xi_{\mathcal{G}^*}(J(m)) = rJ(m)\,.$$

Now we recall that $\xi_{\mathcal{G}^*} = -ad^*_\xi$ (cf [9]) and so we obtain from (5), with $\nu = J(m)$

$$r\|J(m)\|^2 = 0\,.$$

Since m is a point in $\hat{J}^{-1}(\mu) \subset M\backslash J^{-1}(0)$, we have that $\|J(m)\| \neq 0$ and, therefore, $r = 0$. Summarizing, we have thus shown that $\xi_{M_0}(m) \in \mathrm{Ker}(T_m\hat{J})$ if and only if $\xi_{\mathcal{G}^*}(J(m)) = 0$, i.e. $\xi \in \mathcal{G}_{J(m)}$. Note that $J(m) = \|J(m)\|\mu$. The proof of the lemma is then completed by observing that from the definition of the isotropy subalgebras and from the linearity of the fundamental vector fields of the coadjoint action, it readily follows that $\mathcal{G}_{J(m)} = \mathcal{G}_\mu$. □

Assume now that $\mu \in S_{\mathcal{G}^*}$ is a weakly regular value of $\hat{J}$, such that $\hat{M}_\mu = \hat{J}^{-1}(\mu)$ is a submanifold of M_0 and $T_m\hat{M}_\mu = \mathrm{Ker}(T_m\hat{J})$ at each point $m \in \hat{M}_\mu$. From the previous lemma we can deduce that the given action Φ restricts to an action of G_μ on $\hat{M}_\mu$. Put $E_m = T_m(G.m)$ and $E = \bigcup_{m\in\hat{M}_\mu} E_m$ such that $E \subset TM_0|_{\hat{M}_\mu}$. With M_0 now possessing with the Poisson structure induced by the symplectic form ω, we can now state the following proposition, using the terminology of the previous section:

Proposition. If G_μ acts freely on $\hat{M}_\mu$, then the triple $(M_0, \hat{M}_\mu, E)$ is Poisson reducible.

Proof. We have to check that $(M_0, \hat{M}_\mu, E)$ satisfies the conditions (i) to (iv) of section 2.

(i) From lemma 2 we know that $E \bigcap \mathrm{Ker}(T\hat{J}) = E \bigcap T\hat{M}_\mu$ is the sub-bundle of $T\hat{M}_\mu$ with fibres $E_m \bigcap T_m\hat{M}_\mu = T_m(G_\mu.m) = \{\xi_{M_0}(m) \mid \xi \in \mathcal{G}_\mu\}$. This clearly yields an integrable distribution on $\hat{M}_\mu$ (since G_μ acts freely on $\hat{M}_\mu$) and the leaves of the corresponding foliation are the connected components of the G_μ-orbits.

(ii) G_μ is a compact Lie group, being a closed subgroup of the compact Lie group G, and therefore its action on $\hat{M}_\mu$ is proper. Since, by assumption, it is also a free action, the induced foliation on $\hat{M}_\mu$ is regular (see, e.g., [7]). The quotient space $\hat{M}_\mu/G_\mu$ therefore admits a smooth manifold structure such that the projection $\pi_\mu : \hat{M}_\mu \longrightarrow \hat{M}_\mu/G_\mu$ becomes a submersion.

(iii) Let f, g be any two smooth functions on M_0 with differentials vanishing on E, which is equivalent to saying that $\xi_{M_0}(f)(m) = \xi_{M_0}(g)(m) = 0$ for every $\xi \in \mathcal{G}$ and $m \in \hat{M}_\mu$. Since the fundamental vector fields are (global) Hamiltonian vector fields, it follows from the Jacobi identity for Poisson brackets that $\xi_{M_0}(\{f, g\})(m) = 0$ and, hence, $d\{f, g\}$ vanishes on E.

(iv) Let Λ denote the Poisson tensor field on M_0, i.e. $\Lambda = \omega^{-1}$. One easily verifies that $\Lambda_m(E^0_m) = E^\perp_m = T_m(G.m)^\perp$ which, according to (1), yields $\Lambda_m(E^0_m) = \mathrm{Ker}(T_m J)$. In view of lemma 1 we furthermore have that $\mathrm{Ker}(T_m J) \subset \mathrm{Ker}(T_m \hat{J})$ and thus $\Lambda_m(E^0_m) \subset \mathrm{Ker}(T_m \hat{J}) = T_m \hat{M}_\mu$ at each point $m \in \hat{M}_\mu$. Hence, it certainly holds that $\Lambda(E_0) \subset T\hat{M}_\mu + E$, which completes the proof. □

Putting $\mathcal{Q}_\mu = \hat{M}_\mu/G_\mu$, we thus have that for every weakly regular value μ of $\hat{J}$, and assuming G_μ acts freely on $\hat{M}_\mu$, $\mathcal{Q}_\mu$ is a Poisson manifold. Let $i_\mu : \hat{M}_\mu \longrightarrow M_0$ denote the natural inclusion, then the Poisson bracket $\{\ ,\ \}_\mu$ on $\mathcal{Q}_\mu$ is defined in terms of the Poisson bracket on M_0 by

$$\{\tilde{f}, \tilde{g}\}_\mu \circ \pi_\mu = \{f, g\} \circ i_\mu$$

where $\tilde{f}, \tilde{g}$ are any two smooth functions on $\mathcal{Q}_\mu$, and f, g are smooth extensions to M_0 of $\tilde{f} \circ \pi_\mu$ and $\tilde{g} \circ \pi_\mu$, repectively, with $\mathrm{d}f|_E = \mathrm{d}g|_E = 0$.

We now proceed to demonstrate that, under the appropriate regularity assumptions, the reduced space $\mathcal{Q}_\mu$ in fact admits the structure of a *canonical manifold*. Recall that, following Lichnerowicz [19] (see also, e.g., [21]), a canonical manifold is a triple $(P, \{\ ,\ \}, \tau)$ consisting of a Poisson manifold $(P, \{\ ,\ \})$ of constant rank, and a fibration $\tau : P \longrightarrow \mathbb{R}$ such that the connected components of the fibres are the symplectic leaves of the Poisson structure.

Hereafter, $\mu \in S_{\mathcal{G}^*}$ always refers to a (weakly) regular value of $\hat{J}$ with G_μ acting freely on the corresponding level set $\hat{M}_\mu$. We first of all note that $J(\hat{M}_\mu) \subset \rho^{-1}(\mu) = \{k\mu | k \in \mathbb{R}^+_0\}$, with $\mathbb{R}^+_0$ the set of strictly positive real numbers. Moreover, for any $\nu \in J(\hat{M}_\mu)$, $J^{-1}(\nu) \subset \hat{M}_\mu$. From all this it follows that $J|_{\hat{M}_\mu}$ can be regarded as a smooth map from $\hat{M}_\mu$ into $\mathbb{R}^+_0$. Henceforth, we assume that this is a regular map, which, in particular, implies that all level sets $J^{-1}(\nu)$ for $\nu \in J(\hat{M}_\mu)$ are regular co-dimension 1 submanifolds of $\hat{M}_\mu$. Note also that for each $\nu = k\mu$, with $k \neq 0$, $G_\nu = G_\mu$. In view of the Ad^*-equivariance of J, it follows that the G_μ-orbit of a point $m \in \hat{M}_\mu$ is entirely contained in the level set of J passing through m. From this one can then easily deduce that for each $\nu \in J(\hat{M}_\mu)$, the corresponding Marsden–Weinstein reduced space $\mathcal{P}_\nu = J^{-1}(\nu)/G_\nu$ (cf the introduction) is a submanifold of $\mathcal{Q}_\mu$, and the canonical projection of $J^{-1}(\nu)$ onto $\mathcal{P}_\nu$ is simply the restriction of π_μ to $J^{-1}(\nu)$.

It is rather straightforward to verify that, under the given assumptions, the symplectic leaves of the Poisson structure on $\mathcal{Q}_\mu$ are precisely the connected components of these Marsden–Weinstein reduced spaces. This indeed follows from the fact that the characteristic distribution of the Poisson structure on $\mathcal{Q}_\mu$ is generated by the Hamiltonian vector fields on $(\mathcal{Q}_\mu, \{\ ,\ \}_\mu)$ (see, e.g., [9]). These Hamiltonian vector fields are π_μ-related to (the restriction of) Hamiltonian vector fields on M_0 with a G-invariant Hamiltonian, and the latter are known to be tangent to the level sets of the momentum map. From this one can then infer that at each point $\tilde{m} \in \mathcal{Q}_\mu$, the characteristic space of the Poisson structure coincides with the tangent space to the reduced Marsden–Weinstein manifold, say $\mathcal{P}_\nu$, passing through $\tilde{m}$. Moreover, it is not difficult to see that the the symplectic structure on $\mathcal{P}_\nu$, induced by the Poisson structure on $\mathcal{Q}_\mu$, is the same as the one resulting from the Marsden–Weinstein construction.

From the Marsden–Weinstein reduction theory we also know that dim $\mathcal{P}_\nu$ = dim M − dim G − dim G_ν. The previous discussion therefore tells us that the symplectic leaves of $\mathcal{Q}_\mu$ all have the same dimension, namely dim M − dim G − dim G_μ; i.e. $\mathcal{Q}_\mu$ is a Poisson manifold of constant rank. A simple dimensional argument further reveals that dim $\mathcal{Q}_\mu$ = (dim M − dim G − dim G_μ) + 1.

By construction we have that $\|J\|$ is a G-invariant function on M, i.e. $\|J\| \circ \Phi_g = \|J\|$ for each $g \in G$. This in particular implies that $\|J\||_{\hat{M}_\mu}$ induces a well-defined smooth function τ_μ on $\mathcal{Q}_\mu$ such that

$$\tau_\mu \circ \pi_\mu = \|J\||_{\hat{M}_\mu} . \tag{9}$$

Clearly, $\|J\||_{\hat{M}_\mu}$ can be identified with $J|_{\hat{M}_\mu}$ (regarded as a map into $\mathbb{R}_0^+$) and therefore, by assumption, it is a regular map. This can also be checked directly as follows. Let $m \in \hat{M}_\mu$ and $v \in T_m\hat{M}_\mu$. According to lemma 1, $T_mJ(v) = rJ(m)$ for some $r \in \mathbb{R}$. Choosing an orthonormal basis (e^i) of $\mathcal{G}^*$ with respect to the given Euclidean metric on $\mathcal{G}^*$, we can put $J = J_i e^i$, with $J_i \in C^\infty(M)$, and then $\|J\| = (\sum_i J_i{}^2)^{1/2}$. A simple computation then shows us that $d(\|J\||_{\hat{M}_\mu})(m)(v) = r\|J(m)\|$. Observing that $\|J(m)\| \neq 0$ for $m \in \hat{M}_\mu$, and $r \neq 0$ whenever v is not tangent to $J^{-1}(J(m))$, we see that $\|J\||_{\hat{M}_\mu}$ has no singular points. Since π_μ is a submersion it then follows from (9) that τ_μ is a regular map, i.e. $d\tau_\mu(\tilde{m}) \neq 0$ for all $\tilde{m} \in \mathcal{Q}_\mu$. Moreover, for each $k \in \mathrm{Im}(\tau_\mu) \subset \mathbb{R}_0^+$ we readily find that $\tau_\mu^{-1}(k) = J^{-1}(\nu)/G_\nu = \mathcal{P}_\nu$, with $\nu = k\mu$. This finally completes the proof that the triple $(\mathcal{Q}_\mu, \{\ ,\ \}_\mu, \tau_\mu)$ is indeed a canonical manifold.

Summarizing the above analysis, we have thus demonstrated that the following theorem holds.

Theorem 1. Let G be a compact connected Lie group defining a strongly Hamiltonian action on a connected symplectic manifold (M, ω), with Ad*-equivariant momentum map J. Let $\mu \in S_{\mathcal{G}^*}$ be a weakly regular value of the normalized momentum map $\hat{J} = J/\|J\|$ (with respect to a G-invariant metric on $\mathcal{G}^*$). Assume that G_μ acts freely on $\hat{M}_\mu = \hat{J}^{-1}(\mu)$ and that $J|_{\hat{M}_\mu}$ is a regular map. Then, the quotient space $\mathcal{Q}_\mu = \hat{J}^{-1}(\mu)/G_\mu$ admits the structure of a canonical manifold, the symplectic leaves of which are the connected components of the Marsden–Weinstein reduced spaces corresponding to the values of J in $J(\hat{M}_\mu)$. □

Note that the regularity conditions of the theorem are in particular verified when the given action of G on M is a free action, for then both J and $\hat{J}$ are submersive (as can be inferred from (2) and (7)).

4. Application

In this section we briefly indicate how the above geometrical reduction scheme can be applied to a specific class of non-Hamiltonian systems with symmetry, defined on a cotangent bundle. For more details and additional comments we refer the reader to [5].

Let Q be a smooth, finite-dimensional manifold, and T^*Q its cotangent bundle with canonical symplectic form $\omega_Q = \mathrm{d}\theta_Q$, where θ_Q is the Liouville 1-form. In canonical coordinates (q^i, p_i) we have $\theta_Q = p_i \mathrm{d}q^i$. Suppose again that G is a compact, connected Lie group and let $\Phi : G \times T^*Q \longrightarrow T^*Q$ denote the cotangent lift of a smooth left action of G on Q. For simplicity we assume that this action, and therefore also its cotangent lift Φ, is free. It is well known (see, e.g., [9]) that Φ is a strongly Hamiltonian action which, in particular, leaves the Liouville 1-form invariant, i.e. for each $g \in G$

$$\Phi_g^* \theta_Q = \theta_Q . \tag{10}$$

Furthermore, the fundamental vector fields of the action Φ satisfy

$$i_{\xi_{T^*Q}} \omega_Q = -\mathrm{d}J_\xi \tag{11}$$

with $J_\xi = \langle \xi_{T^*Q}, \theta_Q \rangle$.

On T^*Q we now consider a dynamical system X determined by

$$i_X \omega_Q = -\mathrm{d}H - f\theta_Q \tag{12}$$

for some smooth functions $H, f \in C^\infty(T^*Q)$. In canonical coordinates the differential equations corresponding to X read

$$\dot{q}^i = \frac{\partial H}{\partial p_i}(q, p) \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}(q, p) - f(q, p) p_i .$$

The phase space equations of motion of certain mechanical systems with friction can be cast into such a form (cf some models for the Kepler problem with drag [1, 2, 4]). Suppose the given action Φ leaves H invariant, then a straightforward calculation, using (11) and (12), reveals that for each $\xi \in \mathcal{G}$, $X(J_\xi) = -f J_\xi$ and, hence

$$X(J) = -fJ .$$

From this one can first of all deduce that $J^{-1}(0)$ is an invariant submanifold of X. Putting $T^*Q_0 = T^*Q \backslash J^{-1}(0)$ it also follows that the normalized momentum map $\hat{J}$ is a conserved quantity of $X|_{T^*Q_0}$, i.e. $X|_{T^*Q_0}(\hat{J}) = 0$ [5]. All the regular level sets of $\hat{J}$ are therefore invariant submanifolds of the given dynamical system. If we now furthermore assume that also the 'friction coefficient' f is G-invariant, then, taking into account (10) and the invariance of H, one easily finds that Φ is a symmetry of X. The flow of $X|_{J^{-1}(0)}$ then commutes with the induced action of G on $J^{-1}(0)$ and, similarily, for each $\mu \in S_{\mathcal{G}^*}$, the flow of $X|_{\hat{J}^{-1}(\mu)}$ commutes with the action of G_μ induced on $\hat{J}^{-1}(\mu)$. Consequently, the restrictions of X to the invariant submanifolds $J^{-1}(0)$ and $\hat{J}^{-1}(\mu)$ project onto the corresponding quotient spaces $\mathcal{P}_0 = J^{-1}(0)/G$ and $\mathcal{Q}_\mu = \hat{J}^{-1}(\mu)/G_\mu$, respectively. (Recall that we have assumed the given action of the compact Lie group G to be free). Combining all this with the results of the previous section, we can finally state the following reduction theorem for the dynamical system X defined by (12) (cf [5] for details of the proof):

Theorem 2. If both H and f are invariant under the free lifted action Φ of the compact, connected Lie group G on T^*Q, then Φ is a symmetry of X, and it holds that the zero level set of the momentum map J and the level sets of the normalized momentum map $\hat{J}$ are invariant submanifolds of X. Furthermore, we have that (i) the restriction of X to $J^{-1}(0)$ projects onto the reduced symplectic manifold $\mathcal{P}_0$, (ii) the restriction of X to $\hat{J}^{-1}(\mu)$ projects onto the reduced canonical manifold $\mathcal{Q}_\mu$. □

It is well known that, under the given assumptions, the Marsden–Weinstein reduced space $(\mathcal{P}_0, \omega_0)$ is symplectomorphic with $(T^*(Q/G), \omega_{Q/G})$, where Q/G is the orbit space of the action of G on Q [7]. Upon identifying both symplectic manifolds, it is then easily seen that the restriction of X to $J^{-1}(0)$ projects onto the vector field X_0 satisfying the equation

$$i_{X_0}\omega_{Q/G} = -\mathrm{d}H_0 - f_0\theta_{Q/G}$$

with H_0 and f_0 the functions on $T^*(Q/G)$ induced by H and f, respectively, and $\theta_{Q/G}$ the canonical Liouville 1-form on $T^*(Q/G)$.

We now describe in some more detail the reduction of the restriction of X to an invariant submanifold $\hat{J}^{-1}(\mu)$. From (12) we infer that X can be written as

$$X = -\Lambda_Q(\mathrm{d}H) - f\Delta \tag{13}$$

with $\Lambda_Q = \omega_Q^{-1}$ the (canonical) Poisson tensor field on T^*Q and $\Delta = \Lambda_Q(\theta_Q)$ the dilation vector field (i. e. $\Delta = p_i\partial/\partial p_i$). Using the relation $i_\Delta\omega_Q = \theta_Q$ it immediately follows that Δ is invariant under the lifted group action of G on T^*Q and, moreover, that

$$\Delta(J) = J \tag{14}$$

(the components of the momentum map of a lifted group action are homogeneous of the first degree in the canonical momenta p_i). Consequently, following the same reasoning as given above for X, it is seen that $\hat{J}$ is a conserved quantity for Δ and that $\Delta|_{\hat{J}^{-1}(\mu)}$ projects onto $\mathcal{Q}_\mu$. Let us denote this projection by $\tilde{\Delta}_\mu$. Denoting by Λ_μ the induced Poisson tensor field on $\mathcal{Q}_\mu$, it now easily follows from (13) and from the above analysis, that $X|_{\hat{J}^{-1}(\mu)}$ projects onto the vector field

$$\tilde{X}_\mu = -\Lambda_\mu(\mathrm{d}\tilde{H}_\mu) - \tilde{f}_\mu\tilde{\Delta}_\mu$$

where $\tilde{H}_\mu$ and $\tilde{f}_\mu$ are determined by $H|_{\hat{J}^{-1}(\mu)} = \tilde{H}_\mu \circ \pi_\mu$ and $f|_{\hat{J}^{-1}(\mu)} = \tilde{f}_\mu \circ \pi_\mu$, respectively. The reduced system thus consists of a 'Hamiltonian part', $-\Lambda_\mu(\mathrm{d}\tilde{H}_\mu)$, describing the evolution along the symplectic leaves of $\mathcal{Q}_\mu$, and a 'dissipative part', $-\tilde{f}_\mu\tilde{\Delta}_\mu$, which describes the motion transverse to the symplectic leaves. Note that (14) in particular implies that $\Delta(\|J\|) = \|J\|$ from which one immediately deduces that $\tilde{\Delta}_\mu(\tau_\mu) = \tau_\mu$, with τ_μ defined by (9). Hence, $\tilde{X}_\mu$ admits a component transverse to the symplectic foliation of $\mathcal{Q}_\mu$ which, at least locally, can be written as $-\tilde{f}_\mu\tau_\mu\partial/\partial\tau_\mu$.

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